# Solution to 14.pdf

(1) Let $G$ be a finite group of order $n$. Let $\gcd(m, n) = 1$ implies there exists integers $a, b \in \mathbb{Z}$ such that $am + bn = 1$. Let $\phi$ be the map $x \longmapsto x^m$. Consider the map $\xi : G \longrightarrow G$ given by $\xi(x) = x^a$. Then for any $x \in G$, $\phi \circ \xi(x) = \phi(x^a) = x^{am} = x$ since $x = x^1 = x^{am+bn} = x^{am}x^{bn} = x^{am}(x^n)^b = x^{am}$ where $n$ is the order of $G$. Similarly, we have $\xi \circ \phi = \mathrm{Id}_G$. Thus, the $m$-power map is a bijection on $G$.

(2) Given $H \lhd G$, $K \subseteq G$ such that $K$ is closed under group operation and $H \cap K = \{e\}$. We need to show that $HK$ is a subgroup of $G$ iff $K$ is a subgroup of $G$. Suppose $K \leq G$. We first prove that $HK$ is closed under group operation. Let $h_1k_1, h_2k_2 \in HK$. Then, $h_1k_1h_2k_2 = h_1k_1h_2k_1^{-1}k_1k_2 = h_1h_3k_3 \in HK$ where $h_3 = k_1h_2k_1^{-1}$ and $k_3 = k_1k_2$. Now, $e \in HK$ and finally we show that $\alpha = k_1^{-}h_1^{-1}h_2k_2 \in HK$. Since $H \lhd G$, $\alpha \in H$. Since $K \leq G$, $e \in K$ and hence $\alpha = \alpha \cdot e \in HK$. Thus, $HK$ is a subgroup of $G$.

Conversely, suppose $HK \leq G$. Note that $K \subset HK$ since if $k \in K$, $k = e \cdot k \in HK$. But $HK$ is a subgroup implies $k^{-1} \in HK$. We assert that $k^{-1} \in K$. Clearly, $k^{-1} \notin H$ and if $k^{-1} = hk'$ for some $h \in H$ then, $H \cap K \neq \{e\}$. Thus, every element in $K$ has an inverse. As $K$ is closed under group operation, we conclude that $K$ is a subgroup.

(3) We classify all groups of order 12 whose 3 Sylow subgroups are normal. Let $G$ be a group of order $12 = 2^2 \cdot 3$. By third Sylow theorem, if $n_2$ and $n_3$ respectively denote the number of 2-Sylow and 3-Sylow subgroups of $G$ then, $n_2 \in \{1, 3\}$ and $n_3 \in \{1, 4\}$. Let $P$, $Q$ respectively denote a 2-Sylow and a 3-Sylow subgroup of $G$. We assert that $G$ is a semidirect product of $P$ and $Q$. As the orders are relatively prime, we have $P \cap Q = \{1\}$ and $|PQ| = |P||Q|/|P \cap Q| = 2^2 \cdot 3 = 12 = |G|$. Thus, $G = PQ$. Now, the case when 3-Sylow subgroup is normal is the case $n_3 = 1$. This gives $Q \lhd G$ and $G \cong Q \rtimes P$. The 2-Sylow subgroup $P$ has order 4 and $P \cong \mathbb{Z}_4$ or $\mathbb{Z}_2^2$. Similarly, $Q \cong \mathbb{Z}_3$. This gives $G \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_4$ or $G \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_2^2 \cong D_6$ depending on whether the 2-Sylow subgroup is cyclic or not. Further, when $G$ is abelian, we have the possibilities $G \cong \mathbb{Z}_{12}$ or $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.

(4) To show that if $H$ is a Sylow subgroup of $G$ then, $N(N(H)) = N(H)$. One way is clear, $N(N(H)) \supseteq N(H)$. To prove the reverse inclusion let $x \in N(N(H))$ then, $xN(H)x^{-1} = N(H)$. Further, note that $H$ is the unique $p$-Sylow subgroup of $N(H)$ because $nHn^{-1} = H$ for all $n \in N(H)$. We have $xHx^{-1} \subseteq xN(H)x^{-1} = N(H)$, i.e., $xHx^{-1}$ is a $p$-Sylow subgroup of $N(H)$ and hence $xHx^{-1} = H$.

(5) $q$ is a prime number and $p$ divides $q-1$. Let $C_q$ denote the cyclic group of order $q$. Then, $Aut(C_q)$ is a subgroup of $S_q$ with the property that $|Aut(G)| = q-1$. As $p \mid q-1$, by Cauchy's theorem there exists $\phi \in Aut(C_q)$ such that $o(\phi) = p$. Let $H \leqslant Aut(C_q)$ be the cyclic subgroup generated by $\phi$. Then, we have a nontrivial homomorphism from $C_p$ to $Aut(C_q)$ given by $\eta(a) = \phi^a$ for $a \in C_p$. Thus, the semidirect product $G = C_q \rtimes_\eta C_p$ is the required non abelian subgroup of order $pq$ in $S_q$.

(6) $N \trianglelefteq G$ and $H$ is a $p$-Sylow subgroup of $G$ so, $NH$ is a subgroup of $G$. Clearly, $H$ is a subgroup of $NH$ as well and we have $[NH : H] = [N : N \cap H]$ by the isomorphism $NH/H \cong N/N \cap H$. Further, we have $[G : H] = [G : NH][NH : H]$. Consequently, as $H$ is $p$-Sylow subgroup $p \nmid [G : H]$ and hence $p \nmid [NH : H] = [N : N \cap H]$. This proves that $N \cap H$ is $p$-Sylow in $N$.